6.2

Change of variables formula

The substitution formula for functions of one variable is

$$\int_a^b f(x(u))\,\frac{dx}{du}\,du = \int_{x(a)}^{x(b)} f(x)\,dx$$

Here x is expressed in terms of u so we have a function x : R -> R which sends u to an expression x(u), and also a function f : R -> R defined where the variable is called x.

$$\underset{\text{variable } u}{\mathbb{R}} \xrightarrow{\ x\ } \underset{\text{variable } x}{\mathbb{R}} \xrightarrow{\ f\ } \mathbb{R}$$

On the left we are integrating the composite function f ∘ x

Because x probably does not change at the same speed as u, we have to include a factor dx / du

Example:

$$\int_0^2 2x e^{x^2}\,dx. \qquad \text{Put } u = x^2$$

$$du = 2x\,dx$$

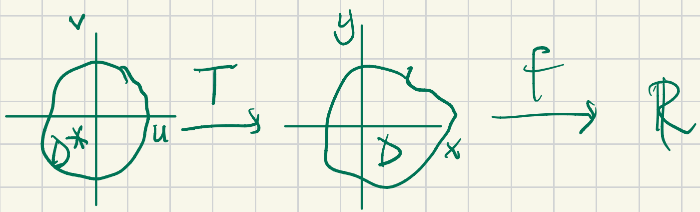
We get $\int_0^4 e^u\,du = e^4 - 1.$

In higher dimensions: the setup is a differentiable map $T : D^* \to D$.
We assume T is 1 - 1 and $D = T(D^*)$. Then

$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

absolute value.



$$T(u,v) = (x(u,v), y(u,v))$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Polar coordinates;

T : D* -> D is

T(r,t) = (x(r,t), y(r,t)) = (r cos t, r sin t).

$$\frac{\partial(x,y)}{\partial(r,t)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

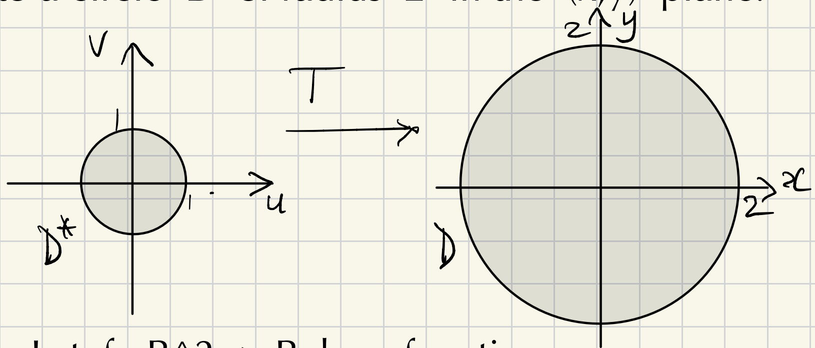
$$= r(\cos^2\theta + \sin^2\theta) = r$$

We get

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(r,\theta)\, r\,dr\,d\theta$$

Pre-class Warm-up!!!

Let T : R^2 -> R^2 be the mapping
T(u,v) = (2u, 2v) and write (x,y) = T(u,v).
T maps the unit circle D* in the (u,v)-plane
to a circle D of radius 2 in the (x,y) plane.



Let f : R^2 -> R be a function.
What is the relationship between

$$A = \iint_D f \, dx\,dy$$

$$B = \iint_{D^*} f \circ T \, du\,dv$$

Which of the following is true?

a. A = 4B

b. A = 2B

c. A = B

d. A = B / 2

e. A = B / 4

Let
Example: $f(x,y) = 1$ for all (x,y)

Then $f \circ T(u,v) = 1$

A = Area of D

B = Area of D*

A = 4B

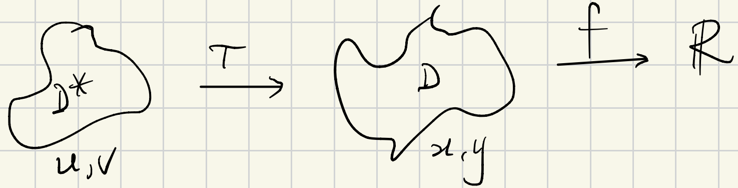
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$$

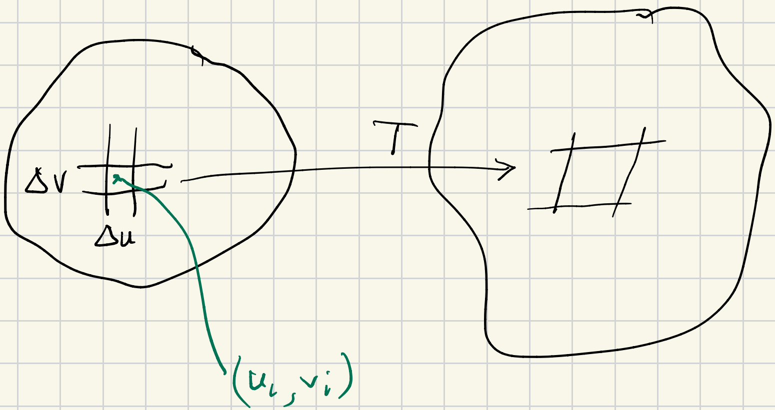
There will be no quiz tomorrow!!!

$$\iint_D f \, dx\,dy = \iint_{D^*} f \circ T \, |4| \, du\,dv.$$

Why does the change of variables formula work?

$$\iint_D f(x,y)\,dx\,dy = \int_{D^*} f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$





$\iint_{D^*} f(u,v)\,du\,dv$ is approximated by Riemann sums

$$\sum \Delta u \Delta v \underbrace{f(u_i, v_i)}_{= f(T(u_i, v_i))}$$

The image of the little u, v square is approximately a parallelogram of area

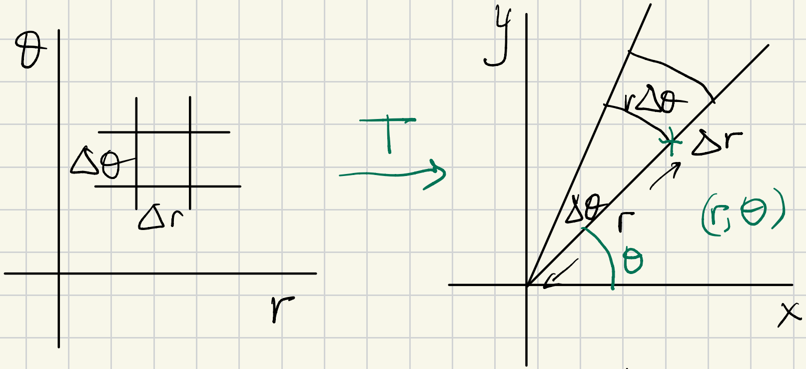
$$\Delta u \Delta v \frac{\partial(x,y)}{\partial(u,v)}$$

$\iint_D f(x,y)\,dx\,dy$ is approx.

$$\sum_{\text{pgm's}} f(T(u_i, v_i))\,\Delta u \Delta v \frac{\partial(x,y)}{\partial(u,v)}$$

which is approx $\int_{D^*} f(u,v) \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$

Polar coordinates



$T(r,\theta) = (r\cos\theta, r\sin\theta)$, $x = r\cos\theta$ $y = r\sin\theta$

$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\, r\,dr\,d\theta.$$

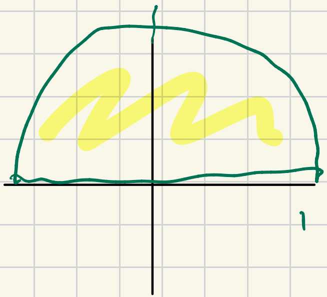
This is explained by:

Area of $T(\Delta r - \Delta\theta$ square$)$

$\simeq r\Delta\theta \cdot \Delta r.$

Example: Find $\iint_D x^2 y \, dx\,dy$

where D is the upper half of the unit disk



In polar coords $0 \le r \le 1$

$0 \le \theta \le \pi$.

$$\iint_D x^2 y \, dx\,dy = \int_0^{\pi}\int_0^1 (r\cos\theta)^2 (r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^1 r^4 \cos^2\theta \sin\theta \, dr\,d\theta = \frac{2}{15}$$

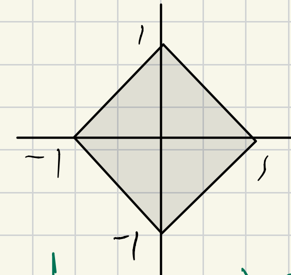
Example. Use the transformation

T(u,v) = ((u+v)/2, (u-v)/2) to find

$x \quad y \quad$ so $u = x+y \quad v = x-y$

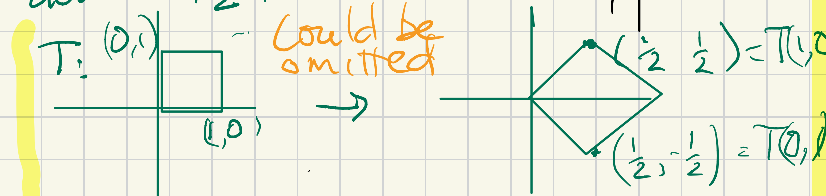
$$\iint_D (x+y)^2 \, dA$$

Where D is the diamond

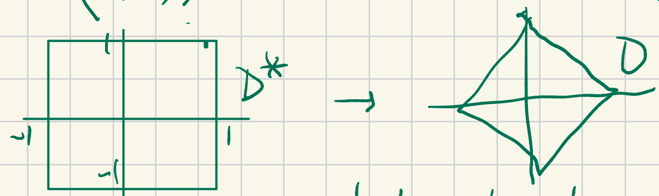


$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$\det = -\frac{1}{2}$.



$T((1,1)) = (1,0) \qquad T((1,-1)) = (0,1)$



$$\iint_D (x+y)^2 \, dA = \int_{-1}^1 \int_{-1}^1 u^2 \left|-\tfrac{1}{2}\right| du\,dv$$

Change of variables for cylindrical coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

Find the volume bounded by
z= √(x^2 + y^2)
And   x^2 + y^2 + z^2 = 1

$$\frac{\pi}{3}(2-\sqrt{2})$$

Change of variables for spherical coordinates.

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \left| \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \right|$$

$$= \left| \cos\phi\left(-\rho^2 \sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) - \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right) \right|$$

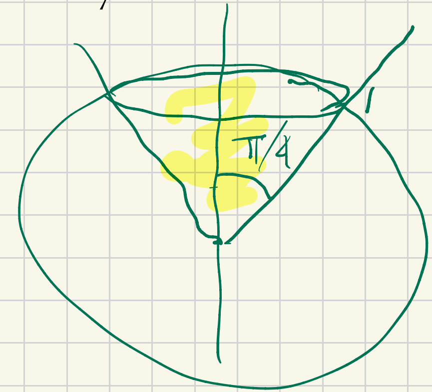
$$= \left| -\rho^2\sin\phi\left(\cos^2\phi(\sin^2\theta+\cos^2\theta) + \sin^2\phi(\sin^2\theta+\cos^2\theta)\right) \right|$$

$$= \left| -\rho^2\sin\phi \right| = \rho^2\sin\phi$$

Thus

$$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Find the volume bounded by
z= √(x^2 + y^2)
and x^2 + y^2 + z^2 = 1



$$\text{Volume } \iiint dx\,dy\,dz$$

$$= \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/4} = \frac{1}{3}\cdot 2\pi \cdot \left(1 - \frac{\sqrt{2}}{2}\right)$$